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Robust Flutter Analysis Considering Mode Shape Variations

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Introduction

ANY aircraft analysis model is subject to simplifications to some extent, leading to uncertainties in the nominal model. Therefore, analysis based on the numerical model generally yields errors in the aerodynamic loads, structural behavior, and critical speeds. Robust flutter analysis usually aims at computing a worst-case flutter speed considering these uncertainties, but can also be used to consider larger variations, such as fuel burn.

In recent years, so-called μ analysis [1] from the control community has been applied to perform robust flutter analysis [2,3]. Most recently, Borglund [4,5] combined classical frequency-domain aeroelasticity with μ analysis to formulate the μ-k method, being closely related to the p-k and g methods [6,7]. In particular, a simple and efficient μ-k algorithm that takes advantage of data from a p-k or g method analysis is described in [8].

At present, the structural uncertainty descriptions available in the literature are based on the assumption of a fixed modal base in the analysis [2,4,9,10]. However, the mode shape variations that result due to variations of structural properties (mass and stiffness distributions) may have a substantial impact on the aeroelastics because they directly influence the unsteady aerodynamic forces.

In this study, it will first be demonstrated that the assumption of a fixed modal base can lead to incorrect flutter results. After that, possible means to account for mode shape variations are presented and evaluated by considering a simple test case.

Robust Flutter Analysis

The basis for robust flutter analysis is the nominal model. In the Laplace domain, the nominal equation of motion for a flexible structure with aerodynamic interaction is given by

$$\mathbf{F}_0(p)v = [p^2\mathbf{M}_0 + (L^2/V^2)\mathbf{K}_0 - (\rho L^2/2)\mathbf{Q}_0(p)]v = 0 \quad (1)$$

where $\mathbf{M}_0$ and $\mathbf{K}_0$ are the mass and stiffness matrices, respectively, and $\mathbf{Q}_0(p)$ is the aerodynamic transfer matrix depending on the nondimensional Laplace variable $p = g + ik$, where g is the damping and k is the reduced frequency. The aerodynamic reference length for computation of the reduced frequency is L, and V and ρ are the airspeed and density, respectively. The vector of displacements in terms of n degrees of freedom of the finite element model is denoted as v. Note that structural damping and the dependence on the Mach number have been omitted for simplicity, but can easily be included. The stated equation is a nonlinear eigenvalue problem that defines a set of eigenvalues p and corresponding eigenvectors v. Nominal stability is guaranteed when all eigenvalues of the nominal system have negative real parts.

In addition to the nominal system, the uncertainty related to this system is defined. In many cases, the uncertain flutter equation depends in a linear fashion on the uncertainty parameters, and can be written in a form defined by a nominal and an uncertain part

$$[\mathbf{F}_0(p) + \mathbf{F}_L(p)\Delta\mathbf{F}_R(p)]v = 0 \quad (2)$$

where $\Delta = \mathrm{diag}(\Delta_S, \Delta_Q)$ with Δ_S and Δ_Q containing structural and aerodynamic uncertainty parameters, respectively. $\mathbf{F}_L$ and $\mathbf{F}_R$ are scaling matrices that determine the impact of the uncertainty parameters on the system, and Δ belongs to a set S_Δ defined as

$$S_\Delta = \{\Delta{:}\Delta \in \mathbf{\Delta} \text{ and } \bar{\sigma}(\Delta) \leq 1\} \quad (3)$$

where $\mathbf{\Delta}$ defines a block structure and $\bar{\sigma}(\cdot)$ denotes the maximum singular value. Premultiplying Eq. (2) by $\mathbf{F}_R\mathbf{F}_0^{-1}$ and defining the system matrix $\mathbf{F}(p) = -\mathbf{F}_R\mathbf{F}_0^{-1}\mathbf{F}_L$ leads to the form

$$[\mathbf{I} - \mathbf{F}(p)\Delta]w = 0 \quad (4)$$

of the uncertain flutter equation, where $\mathbf{I}$ denotes the unitary matrix and $w = \mathbf{F}_R v$ is defined. Robust stability of the system is guaranteed when the system is nominally stable, and when the uncertainty cannot destabilize the system. When the flutter equation is posed in the form of Eq. (4), the μ-k method [4,5,8] can be used to find the airspeed, making a critical eigenvalue $p = ik$ possible for some $\Delta \in S_\Delta$. This airspeed is the worst-case or robust flutter speed.

Mode Shape Variations

Solving Eq. (1) is, in general, computationally expensive, because even simple aircraft structures may require a large number of degrees of freedom n, making the involved matrices very large. A commonly used approach is to perform modal projection of the problem, where only m structural eigenvectors z_j are considered in the modal base $\mathbf{Z} = [z_1 \quad z_2 \quad \cdots \quad z_m]$. It is assumed that the critical flutter mode shape can be represented by a linear combination of those eigenmodes, such that $v = \mathbf{Z}\eta$, giving

$$\mathbf{Z}(\Delta)^T(\mathbf{F}_0 + \mathbf{F}_L\Delta\mathbf{F}_R)\mathbf{Z}(\Delta)\eta = 0 \quad (5)$$

where the $n \times n$ system matrices are reduced to dimension $m \times m$, and η is the modal eigenvector. Typically, m is several orders of magnitude smaller than n.

Fixed Modal Base

Using the common fixed-base approach, it is assumed that the modal base in Eq. (5) is independent of the structural variations, i.e., $\mathbf{Z}(\Delta) = \mathbf{Z}(\Delta = 0) = \mathbf{Z}_0$. The computational effort required to solve the nominal flutter equation is reduced significantly by the modal

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projection. Regarding the uncertain part of the equation, the modal projection will also lead to a reduction of the dimension of Δ. The maximum block dimension for each uncertain parameter δ_j will be reduced from n to m [8]. Because the computational effort using μ analysis depends strongly on the dimension of the uncertainty description, the modal formulation is also beneficial from this point of view.

As known from traditional flutter analysis, modal projection leads to accurate solutions as long as the chosen subset of eigenvectors in $\mathbf{Z}_0$ spans a modal subspace capable of representing the actual flutter mode shape. The flutter mode shape is typically a linear combination of the structural mode shapes corresponding to some of the lower eigenfrequencies, and thus the structural eigenmodes corresponding to the m lowest eigenfrequencies are chosen to build $\mathbf{Z}_0$.

As the structural model is subject to uncertainty and variations, however, the fixed-base approach implies a potential problem. For large structural variations that perturb the structural mode shapes significantly, the nominal modal base may not represent the actual flutter mode shape accurately, leading to incorrect results in the flutter analysis. In the following sections, different approaches to account for mode shape variations are presented.

Perturbed Modal Base

One possibility to account for mode shape variations is to use a Taylor expansion to estimate the perturbed modal base according to

$$\mathbf{Z} \approx \mathbf{Z}_0 + \mathbf{Z}_L \Delta_S \mathbf{Z}_R \tag{6}$$

where $\mathbf{Z}_0$ is the unperturbed modal base, and $\mathbf{Z}_L$ and $\mathbf{Z}_R$ are computed to represent the mode shape variation as a function of the structural variations in $\Delta_S = \mathrm{diag}(\Delta_M, \Delta_K)$. The uncertainty matrices Δ_M and Δ_K contain the uncertainty parameters corresponding to variations in the mass and stiffness properties, respectively. Higher-order terms of the Taylor expansion can be included as well, but would make this approach infeasible, because higher-order terms contain cross derivatives of the modal base with respect to the uncertainty parameters that would increase the problem dimension significantly, even for quite a few uncertainty parameters.

For computation of the first-order coefficients $\mathbf{Z}_L$ and $\mathbf{Z}_R$, an approach as presented in [11] can be followed, where the derivative $z_{i,j}$ of each eigenvector z_i with respect to each structural variation δ_j is computed. Given the derivatives of the mass and stiffness matrices, with respect to the uncertainty parameters,

$$\mathbf{M}_j = \frac{\partial \mathbf{M}}{\partial \delta_j} \quad \text{and} \quad \mathbf{K}_j = \frac{\partial \mathbf{K}}{\partial \delta_j} \tag{7}$$

the eigenvector derivative $z_{i,j}$ is computed by solving the singular system

$$[\mathbf{K} - \omega_i^2 \mathbf{M}] z_{i,j} = -[\mathbf{K}_j - (\omega_i^2)_j \mathbf{M} - \omega_i^2 \mathbf{M}_j] z_i \tag{8}$$

as described in [11], where ω_i is the ith eigenfrequency and $(\omega_i^2)_j = z_i^T(\mathbf{K}_j - \omega_i^2 \mathbf{M}_j) z_i$ is the derivative of its square with respect to δ_j. Finally, the expansion in Eq. (6) is obtained as

$$\mathbf{Z} = \mathbf{Z}_0 + \sum_{j=1}^{r_Z} \mathbf{Z}_{Lj} \Delta_{S_j} \mathbf{Z}_{Rj} = \mathbf{Z}_0 + \mathbf{Z}_L \Delta_S \mathbf{Z}_R \tag{9}$$

where r_Z is the number of uncertainty parameters affecting $\mathbf{Z}$, $\mathbf{Z}_{Lj}$ contains eigenvector derivatives, Δ_{S_j} is a diagonal matrix containing the uncertainty parameter δ_j on the diagonal, and $\mathbf{Z}_{Rj}$ is the unitary matrix. Using this expansion, the flutter equation reads

$$(\mathbf{Z}_0 + \mathbf{Z}_L \Delta_S \mathbf{Z}_R)^T (\mathbf{F}_0 + \mathbf{F}_L \Delta \mathbf{F}_R)(\mathbf{Z}_0 + \mathbf{Z}_L \Delta_S \mathbf{Z}_R)\eta = 0 \tag{10}$$

Using linear fractional transformation matrix operations [12], the uncertain flutter equation can then be posed in the form

$$[\mathbf{I} - \hat{\mathbf{F}}(p)\hat{\Delta}]\hat{w} = \mathbf{0} \tag{11}$$

Note that the higher-order uncertainties $\Delta_S^T \Delta$, $\Delta_S^T \Delta_S$, $\Delta \Delta_S$, and $\Delta_S^T \Delta \Delta_S$ resulting from Eq. (10) are transformed to a linear structured uncertainty block $\hat{\Delta} = \mathrm{diag}(\Delta_S^T, \Delta, \Delta_S)$ of larger dimension. Besides the increased dimension of the uncertainty block, another major drawback is that the favorable upper limit of Δ being in the order of m is no longer valid. In this case, the dimension of the projected problem would exceed the dimension of the full-scale problem, making the modal projection meaningless. The reason for applying it here is to investigate if a linear approximation of the mode shape variation as such is meaningful. If it can be shown that Eq. (10) yields accurate results, a more efficient formulation could possibly be developed.

Updated Modal Base

Another approach to solve Eq. (5) is to apply an iterative solution algorithm, where the flutter equation

$$\mathbf{Z}_i^T (\mathbf{F}_0 + \mathbf{F}_L \Delta \mathbf{F}_R) \mathbf{Z}_i \eta = 0 \tag{12}$$

with fixed $\mathbf{Z}_i$ is solved to determine the worst-case flutter speed. Then, the corresponding worst-case mass and stiffness matrices are computed explicitly for the worst-case perturbation Δ_{S_i}, and an updated modal base $\mathbf{Z}_{i+1}$ is computed by solving the eigenvalue problem

$$[\mathbf{K}(\Delta_{S_i}) + \omega_{i+1}^2 \mathbf{M}(\Delta_{S_i})] z_{i+1} = 0 \tag{13}$$

where $\mathbf{K}(\Delta_{S_i})$ and $\mathbf{M}(\Delta_{S_i})$ are the updated stiffness and mass matrices. The iterations are terminated when the worst-case perturbation has converged according to

$$\|\Delta_{S_{i+1}} - \Delta_{S_i}\|_\infty < \kappa \tag{14}$$

where $\kappa > 0$ is a specified tolerance parameter. In the subsequent case study, the μ-k algorithm described in [8] was used to compute the robust flutter boundary based on Eq. (12). The resulting robust flutter speed and frequency were then used to formulate an optimization problem similar to the one posed in [13] for finding the corresponding worst-case perturbation Δ_i.

Note that, in general, the μ value [1] cannot be evaluated for any block structure $\boldsymbol{\Delta}$, but rather has to be determined by upper and lower bounds. The MATLAB μ Toolbox [14] provides tools for computing these bounds and was used for evaluation of the upper bound in the present study. The resulting largest singular value of the worst-case perturbation Δ_i is the inverse of a lower bound of the corresponding μ value and can thus be used for judging the result of the optimization. In the case of small differences between the upper bound of μ (equal to one for the converged μ-k algorithm) and the computed lower bound, it is likely that the worst-case configuration was found. Because of the nonconvexity of the considered optimization problem, convergence to the global optimum cannot be guaranteed, and thus the optimization was restarted several times from random initial values for higher reliability of the lower bound. In the present case, it was found sufficient to restart the optimization three times to obtain a sufficiently reliable lower bound.

Increased Modal Base

A simple way to account for a potentially insufficient modal base $\mathbf{Z}$ is to increase it by a number of vectors. The larger the number of linearly independent vectors in the modal base, the higher the possibility that the modal base can represent the actual flutter mode shape accurately. Because large modal bases require more computational effort, the number of eigenvectors cannot be increased arbitrarily, but an efficient increase of the modal base is desirable. Several means to increase $\mathbf{Z}$ were investigated in this study.

Additional Eigenvectors

The most straightforward approach is to increase the modal base by additional structural eigenvectors. These eigenvectors establish a base with linearly independent vectors, which is advantageous because it assures that the projected matrices do not become singular due to the projection. The drawback, however, is that the nominal eigenvectors are not related to the structural variations of the system, and adding nominal eigenvectors does not necessarily capture perturbations of the mode shapes due to the uncertainty.

Eigenvector Derivatives

Another possibility to increase the modal base is to compute derivatives $z_{i,j}$ of eigenvectors in the current modal base with respect to structural uncertainties, and to use the derivatives as additional vectors in the modal base. Again, the derivatives can be computed as previously described. These derivatives can be considered as vectors pointing in the direction of the mode shape perturbation due to given uncertainty parameters. Using these derivatives, the modal base is thus increased in the direction of the parametric uncertainty by writing $\mathbf{Z}_{\text{ext}} = [\mathbf{Z} \quad \mathbf{Z}_\Delta]$, where $\mathbf{Z}_\Delta$ contains derivatives of the nominal base. The vectors in $\mathbf{Z}_\Delta$ are, however, not necessarily linearly independent of the existing modal base $\mathbf{Z}$, and they may be similar to each other as well. Besides increasing the problem dimension excessively, including linearly dependent vectors leads to an ill-conditioned problem.

To ensure that only relevant vectors are included, the added vectors were made orthogonal to the existing modal base by using Gram–Schmidt orthogonalization to identify linearly dependent vectors and to find an increased modal base where $z_i^T z_j = 0 \quad \forall\ i \neq j$.

Case Study

As a test case, the wind-tunnel model described in detail in [15] is considered. The 1.2 m semispan model consists of a composite plate that is mounted vertically in the low-speed wind tunnel L2000 at the Royal Institute of Technology as shown in Fig. 1. A beam finite element structural model and doublet-lattice aerodynamics were used for numerical analysis. The aerodynamic model is described in more detail in [4]. To demonstrate the different approaches in a simple way, a variable concentrated mass was put on the leading edge of the wing, at about 40% of the wing span, as shown in the figure. Note that a mass balancing was considered, where only positive mass variations were allowed.

Uncertainty Description

As proposed in [4], linear uncertainties in the system matrices are formulated based on physical reasoning. From an analysis point of

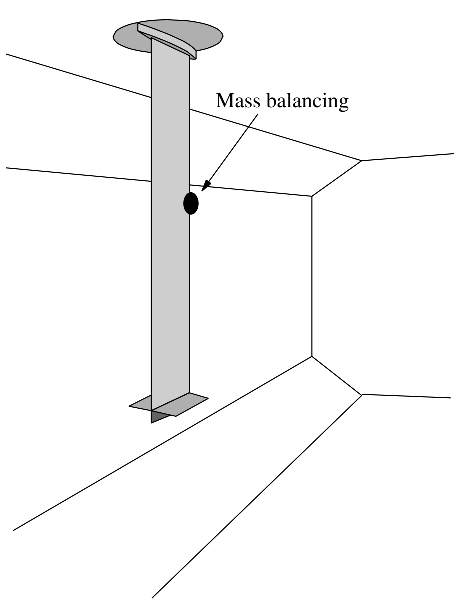

Fig. 1 Wind-tunnel model with leading-edge mass balancing.

view, uncertainties are treated in the same way as variations, for example, fuel level variations. Linear mass variations can, for example, be written in the form

$$\mathbf{M} = \mathbf{M}_0 + \sum_{j=1}^{r_M} \delta_j w_j \mathbf{M}_{\delta j}, \qquad \delta_j \in [-1, 1] \tag{15}$$

where $\mathbf{M}_0$ is the unperturbed mass matrix, $\mathbf{M}_{\delta j}$ is a perturbation matrix due to a variation δ_j, and $w_j > 0$ is the perturbation bound, which may conveniently be included in $\mathbf{M}_{\delta j}$. The number of perturbations of the mass matrix is denoted by r_M. Note that $\mathbf{M}_0$ is not necessarily the nominal mass matrix of the unbalanced wing because, particularly, mass balancing modeling requires a shifted $\mathbf{M}_0$ to obtain the stated boundaries on δ_j. In the present case, the mass balancing was varied from 0 to 1.0 kg. Note that the variation is rather significant, because the wing weight is approximately 1.6 kg without mass balancing.

Minimum-Dimension Description

By rearranging Eq. (15), the uncertain mass matrix can be written

$$\mathbf{M} = \mathbf{M}_0 + \sum_{j=1}^{r_M} \mathbf{M}_{L,j} \Delta_j \mathbf{M}_{R,j} = \mathbf{M}_0 + \mathbf{M}_L \Delta_M \mathbf{M}_R \tag{16}$$

where the matrices $\mathbf{M}_L$ and $\mathbf{M}_R$ can be chosen in different ways. To obtain minimum-dimension uncertainty blocks Δ_j, a singular value decomposition $\mathbf{M}_{\delta,j} = \mathbf{U}_j \mathbf{S}_j \mathbf{V}_j^T$ can be performed, where $\mathbf{S}_j$ is a diagonal matrix with the same rank as the perturbation matrix $\mathbf{M}_{\delta j}$. For each variation parameter δ_j, the corresponding scaling matrices are then chosen as $\mathbf{M}_{L,j} = \mathbf{U}_j \mathbf{S}_j$ and $\mathbf{M}_{R,j} = \omega_j \mathbf{V}_j^T$, and the variation parameter is isolated to a minimum-dimension uncertainty block $\Delta_j = \mathbf{I}\delta_j$, where $\mathbf{I}$ is a unitary matrix of the same dimension as $\mathbf{S}_j$. Thus, the minimum dimension of Δ_j is equal to the rank of the perturbation matrix $\mathbf{M}_{\delta j}$. If a modal formulation is used, the singular value decomposition is applied in the same manner to the projected matrices to reduce computational effort and obtain minimum-dimension uncertainty blocks. Finally, the scaling matrices

$$\mathbf{M}_L = [\mathbf{M}_{L,1} \quad \mathbf{M}_{L,2} \quad \cdots \quad \mathbf{M}_{L,r_M}] \tag{17}$$

$$\mathbf{M}_R = [\mathbf{M}_{R,1}^T \quad \mathbf{M}_{R,2}^T \qquad \mathbf{M}_{R,r_M}^T]^T \tag{18}$$

are assembled along with the structured uncertainty $\Delta_M = \text{diag}(\Delta_1, \Delta_2, \ldots, \Delta_{r_M})$. In the present case, only one single uncertainty parameter was defined to describe the variation of the point-mass balancing.

Aerodynamic Uncertainty

For problems with purely real uncertainties, available solution algorithms for μ require a computational effort that grows exponentially with the dimension of the problem [13]. The problem can be solved more efficiently when some complex-valued variation is introduced. In the most simple case, a small aerodynamic perturbation affecting the aerodynamic loads on all lifting surfaces uniformly can be introduced, which is done in the test case. The considered aerodynamic uncertainty provides enough complex uncertainty to make the problem feasible. In the considered test case, this uncertainty was found to affect the flutter speed just slightly. For the test case, a linear perturbation of the entire aerodynamic matrix $\mathbf{Q}_0$ with a complex uncertainty parameter δ_Q and a real uncertainty bound w_Q was introduced according to

$$\mathbf{Q} = \mathbf{Q}_0(1 + \delta_Q w_Q), \qquad \delta_Q \in [-1, 1] \tag{19}$$

corresponding to the pressure coefficients varying uniformly on the entire wing. There may be more realistic models for describing aerodynamic uncertainty, where the uncertainty depends on the reduced frequency or other parameters. Note, however, that the uncertainty in the present case was only introduced for numerical reasons, and not to capture aerodynamic modeling errors. Minimum-dimension descriptions for the aerodynamic uncertainty are determined in the same way as demonstrated for Δ_M.

Results

For convenient comparison, the results of the robust flutter analysis are summarized in Fig. 2. The figure shows the lower-bound flutter speed as a function of the maximum value of the variable mass. It was found that, due to its position at the leading edge, any mass balancing would increase the flutter speed. The expected solution in all cases was that the most critical perturbation is a zero mass, leading to a robust or worst-case flutter speed equal to the flutter speed of the clean wing at $m_{\max} = 0$. Therefore, a correct robust analysis should result in a worst-case flutter speed independent of the maximum possible mass balancing.

An analysis of the full-scale system without modal projection served as a reference. The full-scale analysis is computationally expensive and can only be performed for very small uncertainty descriptions, such as in the present case. Generally, however, this approach would be infeasible, and the results only serve as a reference for comparison of the different approaches. The performance of the different approaches is judged by the deviation from the full-scale results. The flutter speed of the full-scale model without any uncertainty was found to be 13.9 m/s. As aerodynamic uncertainty is added, the robust flutter speed predicted by the full-scale model is 13.7 m/s (see Fig. 2). The robust flutter speed is independent of $m_{\max}$, because the worst-case perturbation always corresponds to a configuration without mass balancing, due to the stabilizing effect of the added mass. This is correctly predicted by the full-scale model.

Fixed Modal Base

The fixed modal base is the simplest approach and reduces the problem dimension significantly. This approximation is the most computationally efficient. For the case considered, using three eigenmodes in the modal base gives fairly accurate flutter results if no structural uncertainty is present. As the structural uncertainty increases, however, the mode shape variation due to structural variations leads to a flutter mode shape that cannot be represented by the first three eigenmodes any longer, leading to an incorrect flutter speed. For the maximum considered variation, the flutter speed is underpredicted by more than 1 m/s.

Modal Base Perturbation

Modal base perturbation is performed using linear approximations of the mode shape variation. This is the most computationally expensive approach. As shown in Fig. 2, the linear approximation reduces the deviation from the full-scale case, especially for smaller variations compared with the fixed-base case. As the structural variation increases, however, the deviation increases. This is mainly due to the actual mode shape variation not being linear with respect to structural variations. This leads to incorrect mode shape predictions as the variation increases.

An investigation of higher-order expansions of the mode shape with respect to structural perturbations was performed. Results for some representative eigenvectors are shown in Fig. 3. The figure shows, in this specific case, that the first-order estimation overpredicts the mode shape variation due to a fairly large mass balancing. It even deviates more from the true mode shape than the unperturbed mode shape does, implying that the first-order estimation could increase the error. Considering second- and third-order terms improves the estimation, but would increase the dimension of the uncertainty description significantly, making the problem infeasible to solve.

Modal Base Iteration

Modal base iteration combines the small dimension of the uncertainty description from the fixed-base approach with the possibility to account for mode shape variations. The computational

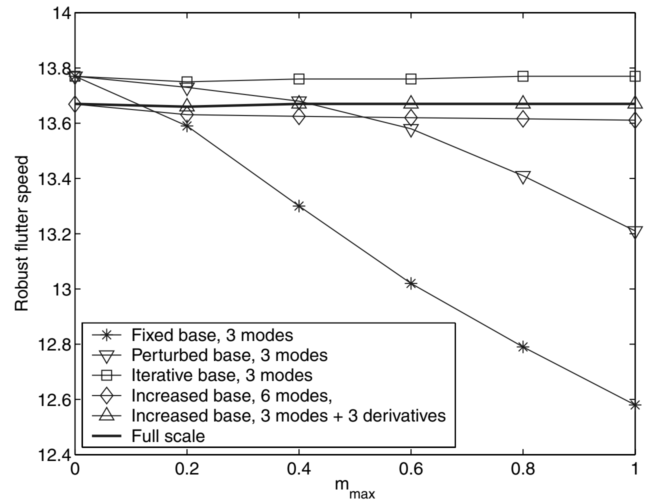

Fig. 2 Comparison of robust flutter speed for different approaches.

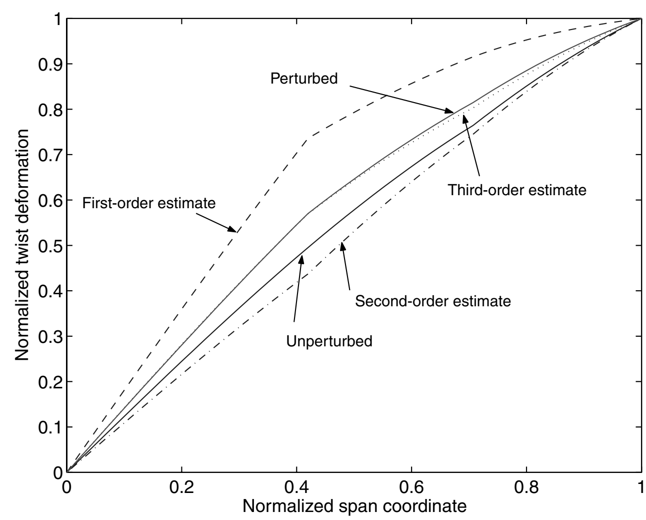

Fig. 3 Perturbation of the third eigenvector due to 0.5 kg mass balancing, and mode shape estimations using Taylor expansions of different order.

effort in this case is about the same as for the fixed-base approximation, multiplied by the number of iterations needed. As shown in Fig. 2, the flutter speed is predicted very close to the flutter speed from the full-scale case. The error is in the order of the deviation due to the modal projection as such. This error is always present because the projection neglects flutter mode shape contributions of higher modes, and it is visible even for zero structural variation. The worst-case perturbation was found to be $\delta_M = -1$, corresponding to the zero mass balancing. The iteration converged after one step, because both the initial and the updated modal base resulted in the same worst-case perturbation and thus fulfilled the convergence criterion in Eq. (14), where a tolerance parameter of $\kappa = 10^{-3}$ was chosen.

Increased Modal Base

Adding three structural eigenvectors to the modal base increases the accuracy significantly, along with a slight increase in computational time. The deviation from the full-scale results is very small in this case. There is, however, a slight trend in the graph indicating that there may be some deviation for increasing variations.

Adding orthogonalized eigenvector derivatives to an orthogonalized modal base yields very accurate results. In the case presented in Fig. 2, first-order derivatives were computed for the three nominal eigenvectors, leading to an additional set of three vectors. Note that there is no solution for the case $m_{\max} = 0$ in Fig. 2, because there exist no eigenvector derivatives for this value of the mass.

Conclusions

The objective of this study was to evaluate different approaches for taking mode shape variations into account in robust flutter analysis. In general, when dealing with small structural variations, the fixed modal base is sufficiently accurate. Even though structural uncertainties as such can have great impact on the flutter behavior, the impact due to mode shape variations appears to be a secondary effect. For large variations, however, the mode shape variation may lead to considerable errors in the prediction of the flutter speed.

A first-order Taylor expansion of the modal base is not very useful, because in cases with small perturbations, mode shape variations can be neglected, and in cases with larger perturbations, the linear approximation can be even less representative than the unperturbed modal base. Higher-order Taylor expansions for the mode shape variation, on the other hand, lead to a substantial increase of the dimension of the uncertainty description, making the problem infeasible to solve.

Two useful methods for taking mode shape variation into account were provided. First, the modal base may be increased by deriving an appropriate set of additional vectors to be included in the modal base. In the most simple case, adding structural eigenvectors can improve the results, but there is no guarantee that these eigenvectors capture the effect of the structural variation. It was shown that the robust flutter results can be improved significantly by including mode shape derivatives with respect to the structural variation.

Second, the iterative approach was found useful for capturing mode shape variations without the need of mode shape derivatives. The main advantage with this approach is that a small uncertainty description is conserved in each iteration. A drawback is that computational time increases as several iterations have to be performed. Further, it is essential that the global optimum to the underlying optimization problem is found if the iterations should converge.

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